

VIRASORO ALGEBRA IN LÖWNER-KUFAREV CONTOUR DYNAMICS

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ABSTRACT. Contour dynamics is a classical subject both in physics and in complex analysis. We show that the dynamics provided by the Löwner-Kufarev ODE and PDE possesses a rigid algebraic structure given by the Virasoro algebra. Namely, the ‘positive’ Virasoro generators span the holomorphic part of the complexified vector bundle over the space of univalent functions, smooth on the boundary. In the covariant formulation they are conserved by the Löwner-Kufarev evolution. The ‘negative’ Virasoro generators span the antiholomorphic part. They contain a conserved term and we give an iterative method to obtain them based on the Poisson structure of the Löwner-Kufarev evolution. The Löwner-Kufarev PDE provides a distribution of the tangent bundle of non-normalized univalent functions, which forms the tangent bundle of normalized ones. It also gives an explicit correspondence between the latter bundle and the holomorphic eigen space of the complexified Lie algebra of vector fields on the unit circle. Finally, we give Hamiltonian and Lagrangian formulations of the motion within the coefficient body in the field of an elliptic operator constructed by means of Virasoro generators. We also discuss relations between CFT and SLE.

1. INTRODUCTION

The challenge of structural understanding of non-equilibrium interface dynamics has become increasingly important in mathematics and physics. Dynamical interfacial properties, such as fluctuations, nucleation and aggregation, mass and charge transport, are often very complex. There exists no single theory or model that can predict all such properties. Many physical processes, as well as complex dynamical systems, iterations and construction of Lie semigroups with respect to the composition operation, lead to the study of growing systems of plane domains. Recently, it has become clear that one-parameter expanding evolution families of simply connected domains in the complex plane in some special models has been governed by infinite systems of evolution parameters, conservation laws. This phenomenon reveals a bridge between a non-linear evolution of complex shapes emerged in physical problems, dissipative in most of the cases, and exactly solvable models. A sample problem is the Laplacian growth, in which the harmonic (Richardson’s) moments

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are conserved under the evolution, see e.g., [23, 33]. The infinite number of evolution parameters reflects the infinite number of degrees of freedom of the system, and clearly suggests to apply field theory methods as a natural tool of study. The Virasoro algebra provides a structural background in most of field theories, and it is not surprising that it appears in soliton-like problems, e.g., KdV or Toda hierarchies, see [6, 10].

Another group of models, in which the evolution is governed by an infinite number of parameters, can be observed in controllable dynamical systems, where the infinite number of degrees of freedom follows from the infinite number of driving terms. Surprisingly, the same structural background appears again for this group. We develop this viewpoint in the present paper.

One of the general approaches to the growing contour evolution was provided by Löwner and Kufarev [20, 26]. The contour evolution is described by a time-dependent conformal parametric map from a canonical domain, the unit disk in most of the cases, onto the domain bounded by the contour for each fixed instant. In fact, these one-parameter conformal maps satisfy the Löwner-Kufarev partial differential equation. A characteristic equation to this PDE represents an infinite dimensional controllable system for which the infinite number of conservation laws is given by the Virasoro generators in their covariant form.

Recently, Friedrich and Werner [8], and independently Bauer and Bernard [4], found relations between SLE (stochastic or Schramm-Löwner evolution) and the highest weight representation of the Virasoro algebra.

All above results encouraged us to conclude that the Virasoro algebra is a common structural basis for these and possibly other types of contour dynamics and we present the development in this direction here. For the first time, a construction, which appeared in the field theory plays the algebraic structural background for the contour evolution in classical complex analysis.

The structure of the paper is as follows. Sections 2 and 3 contain the necessary background on the Virasoro algebra and the Löwner-Kufarev equations. The main results are contained in Sections 4 and 5. In Section 4 we construct the Poisson structure on the cotangent bundle of the space of univalent functions smooth on the boundary and the Hamiltonian system generated by the Löwner-Kufarev equation in ordinary derivatives. We establish that the holomorphic Virasoro generators in the covariant formulation are conserved under the Löwner-Kufarev evolution (Theorem 2). The antiholomorphic generators are proved to contain a conserved term and we give an iterative method to obtain them based on the Poisson structure of the Löwner-Kufarev evolution. The Löwner-Kufarev PDE is shown to provide a distribution of the tangent bundle of non-normalized univalent functions, which forms the tangent bundle of normalized ones. It also gives an explicit correspondence between the latter bundle and the holomorphic eigen space of the complexified Lie algebra of vector fields on the unit circle. In Section 5, we give Hamiltonian and Lagrangian formulations of the motion within the coefficient body in the field of an elliptic operator constructed by means of Virasoro generators. The solutions with constant velocity coordinates are found. We prove that the norm of the driving function in the

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The Virasoro algebra Vir plays a prominent role in modern mathematical physics, both in field theories and solvable models. It appears in physics literature as an algebra obeyed by the stress-energy tensor and associated with the conformal group, the Virasoro-Bott group, of the worldsheet in two dimensions, see e.g., [24]. It is a unique central extension of the Lie algebra for the Lie-Fréchet group $\text{Diff } S^1$ of sense-preserving diffeomorphisms of the unit circle S^1 , and it is an infinite-dimensional real vector space. The extension is characterized by a real parameter c , so the Virasoro algebra refers to a class of isomorphic Lie algebras corresponding to different values of c . At the same time the Virasoro algebra is intrinsically related to the KdV canonical structure where the Virasoro brackets become the Magri brackets for the Miura transformations of elements of the phase space of the KdV hierarchy (see, e.g., [6, 10]).

$$\{L_m, L_n\}_{Vir} = (n - m)L_{m+n} + \frac{c}{12}n(n^2 - 1)\delta_{n,-m}, \quad \{L_n, c\}_{Vir} = 0, \quad n, m \in \mathbb{Z},$$

Mathematically, the Virasoro algebra appeared for the first time as a central extension by the *Gelfand-Fuchs cocycle* [9] of the Lie algebra $\text{Vect } S^1$ of smooth vector fields $\phi \frac{d}{db}$ on the unit circle S^1 (see [9]), where the Lie bracket is defined to be the commutator of vector fields

$$(1) \quad [\phi_1, \phi_2] = \phi_1 \phi_2' - \phi_2 \phi_1'.$$

Each element of the Lie-Fréchet group $\text{Diff } S^1$ is represented as $z = e^{i\alpha(\theta)}$ with a monotone increasing C^∞ real-valued function $\alpha(\theta)$, such that $\alpha(\theta + 2\pi) = \alpha(\theta) + 2\pi$. The Lie algebra for this group is identified with $\text{Vect } S^1$. The relation of this Lie algebra to $\text{Diff } S^1$ is subtle because the exponential map is not even locally a homeomorphism.

2.1. Canonical identification. The entire necessary background of unitary representations of $\text{Diff } S^1$ is found in the study of Kirillov's homogeneous Kählerian manifold $\text{Diff } S^1/S^1$. We deal with the analytic representation of $\text{Diff } S^1/S^1$. Let \mathbf{S} stand for the whole class of univalent functions f in the unit disk U normalized by $f(z) = z(1 + \sum_{n=1}^{\infty} c_n z^n)$ about the origin and C^∞ -smooth on the boundary S^1 of U . Given a map $f \in \mathbf{S}$ we construct the adjoint univalent meromorphic map

$$g(z) = d_1 z + d_0 + \frac{d_{-1}}{z} + \dots,$$

defined in the exterior $U^* = \{z : |z| > 1\}$ of U , and such that $\hat{\mathbb{C}} \setminus \overline{f(U)} = g(U^*)$. Both functions are extendable onto S^1 . This conformal welding gives the identification of the homogeneous manifold $\text{Diff } S^1/S^1$ with the space \mathbf{S} : $\mathbf{S} \ni f \leftrightarrow f^{-1} \circ g|_{S^1} \in \text{Diff } S^1/S^1$, or with the smooth contours $\Gamma = f(S^1)$ that enclose univalent domains Ω of conformal radius 1 with respect to the origin and such that $\infty \notin \Omega$, $0 \in \Omega$, see [1], [13]. So one can construct complexification of $\text{Vect } S^1$ and further projection of the holomorphic part to the set $\mathcal{M} \subset \mathbb{C}^\mathbb{N}$, which is the projective limit of the coefficient bodies $\mathcal{M} = \lim_{n \leftarrow \infty} \mathcal{M}_n$, where

$$(2) \quad \mathcal{M}_n = \{(c_1, \dots, c_n) : f \in \mathbf{S}\}.$$

The holomorphic Virasoro generators can then be realized by the first order differential operators

$$L_j = \partial_j + \sum_{k=1}^{\infty} (k+1) c_k \partial_{j+k}, \quad j \in \mathbb{N},$$

in terms of the affine coordinates of \mathcal{M} , acting over the set of holomorphic functions, where $\partial_k = \partial/\partial c_k$. We explain the details in the next subsection.

2.2. Complexification. Let us introduce local coordinates on the manifold $\mathcal{M} = \text{Diff } S^1/S^1$ in the concordance with the local coordinates on the space \mathbf{S} of univalent functions smooth on the boundary. Observe that \mathcal{M} is a real infinite-dimensional manifold, whereas \mathbf{S} is a complex manifold. We are aimed at a complexification of $T\mathcal{M}$ which admits a holomorphic projection to $T\mathbf{S}$, where $\text{Vect } {}_0S^1 = \text{Vect } S^1/\text{const}$ is a module over the ring of smooth functions, which is associated with the tangent bundle $T\mathcal{M}$.

Given a real vector space V the complexification $V_{\mathbb{C}}$ is defined as the tensor product with the complex numbers $V \otimes_{\mathbb{R}} \mathbb{C}$. Elements of $V_{\mathbb{C}}$ are of the form $v \otimes z$. In addition, the vector space $V_{\mathbb{C}}$ is a complex vector space that follows by defining multiplication by complex numbers, $\alpha(v \otimes z) = v \otimes \alpha z$ for complex α and z and $v \in V$. The space V is naturally embedded into $V \otimes \mathbb{C}$ by identifying V with $V \otimes 1$. Conjugation is defined by introducing a canonical conjugation map on $V_{\mathbb{C}}$ as $\overline{v \otimes z} = v \otimes \bar{z}$.

An almost complex structure J on V can be extended by linearity to the complex structure J on $V_{\mathbb{C}}$ by $J(v \otimes z) = J(v) \otimes z$. Observe that

$$\overline{J(v \otimes z)} = J(\overline{v \otimes z}).$$

Eigenvalues of extended J are $\pm i$, and there are two eigenspaces $V^{(1,0)}$ and $V^{(0,1)}$ corresponding to them given by projecting $\frac{1}{2}(1 \mp iJ)v$. $V_{\mathbb{C}}$ is decomposed into the direct sum $V_{\mathbb{C}} = V^{(1,0)} \oplus V^{(0,1)}$, where $V^{(1,0)} = \{v \otimes 1 - J(v) \otimes i \mid v \in V\}$ and $V^{(0,1)} = \{v \otimes 1 + J(v) \otimes i \mid v \in V\}$ are the eigen spaces corresponding to $\pm i$.

An almost complex structure on $\text{Vect } {}_0S^1$ may be defined as follows (see [1]). We identify $\text{Vect } {}_0S^1$ with the functions with vanishing mean value over S^1 . It gives

$$\phi(\theta) = \sum_{n=1}^{\infty} a_n \cos n\theta + b_n \sin n\theta.$$

Let us define an almost complex structure by the operator

$$(3) \quad J(\phi)(\theta) = \sum_{n=1}^{\infty} -a_n \sin n\theta + b_n \cos n\theta.$$

On $\text{Vect } {}_0S^1 \otimes \mathbb{C}$, the operator J diagonalizes and we have the identification

$$\text{Vect } {}_0S^1 \ni \phi \leftrightarrow v := \frac{1}{2}(\phi - iJ(\phi)) = \sum_{n=1}^{\infty} (a_n - ib_n)e^{in\theta} \in (\text{Vect } {}_0S^1 \otimes \mathbb{C})^{(1,0)},$$

and the latter extends into the unit disk as a holomorphic function.

The Kirillov infinitesimal action [12] of $\text{Vect } {}_0S^1$ on \mathbf{S} is given by a variational formula due to Schaeffer and Spencer [30, page 32] which lifts the actions from the Lie algebra $\text{Vect } {}_0S^1$ onto \mathbf{S} . Let $f \in \mathbf{S}$ and let $\phi(e^{i\theta}) := \phi(\theta) \in \text{Vect } {}_0S^1$ be a C^∞ real-valued function in $\theta \in (0, 2\pi]$. The infinitesimal action $\theta \mapsto \theta + \varepsilon\phi(e^{i\theta})$ yields a variation of the univalent function $f^*(z) = f + \varepsilon \delta_v f(z) + o(\varepsilon)$, where

$$(4) \quad \delta_v f(z) = \frac{f^2(z)}{2\pi} \int_{S^1} \left(\frac{wf'(w)}{f(w)} \right)^2 \frac{v(w)dw}{w(f(w) - f(z))},$$

and $\phi \leftrightarrow v$ by the above identification. Kirillov and Yuriev [13], [14] (see also [1]) established that the variations $\delta_\phi f(\zeta)$ are closed with respect to the commutator (1), and the induced Lie algebra is the same as $\text{Vect } {}_0S^1$. The Schaeffer-Spencer operator is linear.

Treating $T\mathcal{M}$ as a real vector space, the operator δ_ϕ transfers the complex structure J from $\text{Vect } {}_0S^1$ to $T\mathcal{M}$ by $J(\delta_\phi) := \delta_{J(\phi)}$. By abuse of notation, we denote the new complex structure on $T\mathcal{M}$ by the same character J . Then it splits the complexification $T\mathcal{M}_{\mathbb{C}}$ into two eigenspaces $T\mathcal{M}_{\mathbb{C}} = T\mathcal{M}^{(1,0)} \oplus T\mathcal{M}^{(0,1)}$. Therefore, $\delta_v = \delta_{\phi - iJ(\phi)} := \delta_\phi - iJ(\delta_\phi) \in T\mathcal{M}^{(1,0)}$. Observe that $2z\partial_z = -i\partial_\theta$ on the unit circle $z = e^{i\theta}$, and $L_k = z^{k+1}d/dz = -\frac{1}{2}ie^{ik\theta}d/d\theta$ on S^1 . Let us take the basis of $\text{Vect } {}_0S^1 \otimes \mathbb{C}$ in the form $\nu_k = -ie^{ik\theta}$ in order to keep the index of vector fields the same as for L_k . Then, the commutator satisfies the Witt relation $\{\nu_m, \nu_n\} = (n - m)\nu_{n+m}$. Taking elements $\nu_k = -iw^k$, $|w| = 1$ in the integrand of (4) we calculate the residue in (4) and obtain so called Kirillov operators

$$L_j[f](z) = \delta_{\nu_j} f(z) = z^{j+1} f'(z), \quad j = 1, 2, \dots,$$

so that these L_j are the holomorphic coordinates on $T\mathcal{M}^{(1,0)}$. In terms of the affine coordinates in \mathcal{M} we get the Kirillov operators as

$$L_j = \partial_j + \sum_{k=1}^{\infty} (k+1)c_k \partial_{j+k},$$

where $\partial_k = \partial/\partial c_k$. They satisfy the Witt commutation relation

$$\{L_m, L_n\} = (n-m)L_{n+m}.$$

For $k = 0$ we obtain the operator L_0 , which corresponds to the constant vectors from $\text{Vect } S^1$, $L_0[f](z) = zf'(z) - f(z)$. The elements of the Fourier basis $-ie^{-i\theta k}$ with negative indices (corresponding to $T\mathcal{M}^{(0,1)}$) are extended into U by $-iz^{-k}$. Substituting them in (4) we get very complex formulas for L_{-k} , which functionally depend on L_k (see [1], [12]), and which are dual to L_k with respect to the action of J . The first two operators are calculated as

$$\begin{aligned} L_{-1}[f](z) &= f'(z) - 2c_1 f(z) - 1, \\ L_{-2}[f](z) &= \frac{f'(z)}{z} - \frac{1}{f(z)} - 3c_1 + (c_1^2 - 4c_2)f(z), \end{aligned}$$

see [14].

This procedure gives a nice links between representations of the Virasoro algebra and the theory of univalent functions. The Löwner-Kufarev equations proved to be a powerful tool to work with univalent functions (the famous Bieberbach conjecture was proved [5] using Löwner method). In the following section we show how Löwner-Kufarev equations can be used in a representation of the Virasoro algebra. In particular, we identify $T\mathcal{M}^{(1,0)}$ with $T\mathcal{M}$, equipped with its natural complex structure given by coefficients of univalent functions, by means the Löwner-Kufarev PDE.

3. LÖWNER-KUFAREV EQUATIONS

A time-parameter family $\Omega(t)$ of simply connected hyperbolic univalent domains forms a *Löwner subordination chain* in the complex plane \mathbb{C} , for $0 \leq t < \tau$ (where τ may be ∞), if $\Omega(t) \subsetneq \Omega(s)$, whenever $t < s$. We suppose that the origin is an interior point of the Carathéodory kernel of $\{\Omega(t)\}_{t=0}^{\tau}$.

A Löwner subordination chain $\Omega(t)$ is described by a time-dependent family of conformal maps $z = f(\zeta, t)$ from the unit disk $U = \{\zeta : |\zeta| < 1\}$ onto $\Omega(t)$, normalized by $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2 + \dots$, $a_1(t) > 0$, $\dot{a}_1(t) > 0$. After Löwner's 1923 seminal paper [20] a fundamental contribution to the theory of Löwner chains was made by Pommerenke [25, 26] who described governing evolution equations in partial and ordinary derivatives, known now as the Löwner-Kufarev equations due to Kufarev's work [16].

One can normalize the growth of evolution of a subordination chain by the conformal radius of $\Omega(t)$ with respect to the origin by $a_1(t) = e^t$.

Löwner [20] studied a time-parameter semigroup of conformal one-slit maps of the unit disk U arriving then at an evolution equation called after him. His main

achievement was an infinitesimal description of the semi-flow of such maps by the Schwarz kernel that led him to the Löwner equation. This crucial result was then generalized in several ways (see [26] and the references therein).

We say that the function p is from the Carathéodory class if it is analytic in U , normalized as $p(\zeta) = 1 + p_1\zeta + p_2\zeta^2 + \dots$, $\zeta \in U$, and such that $\operatorname{Re} p(\zeta) > 0$ in U . Pommerenke [25, 26] proved that given a subordination chain of domains $\Omega(t)$ defined for $t \in [0, \tau)$, there exists a function $p(\zeta, t)$, measurable in $t \in [0, \tau)$ for any fixed $z \in U$, and from the Carathéodory class for almost all $t \in [0, \tau)$, such that the conformal mapping $f : U \rightarrow \Omega(t)$ solves the equation

$$(5) \quad \frac{\partial f(\zeta, t)}{\partial t} = \zeta \frac{\partial f(\zeta, t)}{\partial \zeta} p(\zeta, t),$$

for $\zeta \in U$ and for almost all $t \in [0, \tau)$. The equation (5) is called the Löwner-Kufarev equation due to two seminal papers: by Löwner [20] who considered the case when

$$(6) \quad p(\zeta, t) = \frac{e^{iu(t)} + \zeta}{e^{iu(t)} - \zeta},$$

where $u(t)$ is a continuous function regarding to $t \in [0, \tau)$, and by Kufarev [16] who proved differentiability of f in t for all ζ from the kernel of $\{\Omega(t)\}$ in the case of general p in the Carathéodory class.

Let us consider a reverse process. We are given an initial domain $\Omega(0) \equiv \Omega_0$ (and therefore, the initial mapping $f(\zeta, 0) \equiv f_0(\zeta)$), and a function $p(\zeta, t)$ of positive real part normalized by $p(\zeta, t) = 1 + p_1\zeta + \dots$. Let us solve the equation (5) and ask ourselves, whether the solution $f(\zeta, t)$ defines a subordination chain of simply connected univalent domains $f(U, t)$. The initial condition $f(\zeta, 0) = f_0(\zeta)$ is not given on the characteristics of the partial differential equation (5), hence the solution exists and is unique but not necessarily univalent. Assuming s as a parameter along the characteristics we have

$$\frac{dt}{ds} = 1, \quad \frac{d\zeta}{ds} = -\zeta p(\zeta, t), \quad \frac{df}{ds} = 0,$$

with the initial conditions $t(0) = 0$, $\zeta(0) = z$, $f(\zeta, 0) = f_0(\zeta)$, where z is in U . Obviously, $t = s$. Observe that the domain of ζ is the entire unit disk. However, the solutions to the second equation of the characteristic system range within the unit disk but do not fill it. Therefore, introducing another letter w (in order to distinguish the function $w(z, t)$ from the variable ζ) we arrive at the Cauchy problem for the Löwner-Kufarev equation in ordinary derivatives

$$(7) \quad \frac{dw}{dt} = -wp(w, t),$$

for a function $\zeta = w(z, t)$ with the initial condition $w(z, 0) = z$. The equation (7) is a non-trivial characteristic equation for (5). Unfortunately, this approach requires the extension of $f_0(w^{-1}(\zeta, t))$ into the whole U (w^{-1} means the inverse function) because the solution to (5) is the function $f(\zeta, t)$ given as $f_0(w^{-1}(\zeta, t))$, where $\zeta = w(z, s)$ is a solution of the initial value problem for the characteristic equation (7) that maps

U into U . Therefore, the solution of the initial value problem for the equation (5) may be non-univalent.

Solutions to the equation (7) are holomorphic univalent functions $w(z, t) = e^{-t}z + a_2(t)z^2 + \dots$ in the unit disk that map U into itself. Every function f from the class \mathbf{S} can be represented by the limit

$$(8) \quad f(z) = \lim_{t \rightarrow \infty} e^t w(z, t),$$

where $w(z, t)$ is a solution to (7) with some function $p(z, t)$ of positive real part for almost all $t \geq 0$ (see [26, pages 159–163]). Each function $p(z, t)$ generates a unique function from the class \mathbf{S} . The reciprocal statement is not true. In general, a function $f \in \mathbf{S}$ can be obtained using different functions $p(\cdot, t)$.

Now we are ready to formulate the condition of univalence of the solution to the equation (5), which can be obtained by combination of known results of [26].

Theorem 1. [26, 27] *Given a function $p(\zeta, t)$ of positive real part normalized by $p(\zeta, t) = 1 + p_1\zeta + \dots$, the solution to the equation (5) is unique, analytic and univalent with respect to ζ for almost all $t \geq 0$, if and only if, the initial condition $f_0(\zeta)$ is taken in the form (8), where the function $w(\zeta, t)$ is the solution to the equation (7) with the same driving function p .*

Recently, we started to look at Löwner-Kufarev equations from the point of view of motion in the space of univalent functions where Hamiltonian and Lagrangian formalisms play a central role (see, [32]). Some connections with the Virasoro algebra were also observed in [22, 32]. The present paper generalizes these attempts and gives their closed form. The main conclusion is that the Löwner-Kufarev equations are naturally linked to the holomorphic part of the Virasoro algebra. Taking holomorphic Virasoro generators L_n as a basis of the tangent space to the coefficient body for univalent functions at a fixed point, we see that the driving function in the Löwner-Kufarev theory generates generalized moments for motions within the space of univalent functions. Its norm represents the energy of this motion. The holomorphic Virasoro generators in their co-tangent form will become conserved quantities of the Löwner-Kufarev ODE. The Löwner-Kufarev PDE becomes a transition formula from the affine basis to Kirillov's basis of the holomorphic part of the complexified tangent space to \mathcal{M} at any point. Finally, we propose to study an alternate Löwner-Kufarev evolution instead of subordination.

4. WITT ALGEBRA AND THE CLASSICAL LÖWNER-KUFAREV EQUATIONS

In the following subsections we reveal the structural role of the Witt algebra as a background of the classical Löwner-Kufarev contour evolution. As we see further, the conformal anomaly and the Virasoro algebra appear as a quantum or stochastic effect in SLE.

4.1. Löwner-Kufarev ODE. Let us consider the functions

$$w(z, t) = e^{-t}z \left(1 + \sum_{n=1}^{\infty} c_n(t)z^n \right),$$

satisfying the Löwner-Kufarev ODE

$$(9) \quad \frac{dw}{dt} = -wp(w, t),$$

with the initial condition $w(z, 0) = z$, and with the function $p(z, t) = 1 + p_1(t)z + \dots$ which is holomorphic in U and measurable with respect to $t \in [0, \infty)$, such that $\operatorname{Re} p > 0$ in U . The function $w(z, t)$ is univalent and maps U into U .

Lemma 1. *Let the function $w(z, t)$ be a solution to the Cauchy problem for the equation (9) with the initial condition $w(z, 0) = z$. If the driving function $p(\cdot, t)$, being from the Carathéodory class for almost all $t \geq 0$, is C^∞ smooth in the closure \hat{U} of the unit disk U and summable with respect to t , then the boundaries of the domains $B(t) = w(U, t) \subset U$ are smooth for all t .*

Proof. Observe that the continuous and differentiable dependence of the solution to a differential equation $\dot{x} = F(t, x)$ on the initial condition $x(0) = x_0$ is a classical problem. One can refer, e.g., to [35] in order to assure that summability of $F(\cdot, x)$ regarding to t for each fixed x and continuous differentiability (C^1 with respect to x for almost all t) imply that the solution $x(t, x_0)$ exists, is unique, and is C^1 with respect to x_0 . In our case, the solution to (9) exists, is unique, analytic in U , and moreover, C^1 on its boundary S^1 . Let us differentiate (9) inside the unit disk U with respect to z and write

$$\log w' = - \int_0^t (p(w(z, \tau), \tau) + w(z, \tau)p'(w(z, \tau), \tau)) d\tau,$$

choosing the branch of the logarithm such as $\log w'(0, t) = -t$. This equality is extendable onto S^1 because the right-hand side is, and therefore, w' is C^1 and w is C^2 on S^1 . We continue analogously and write the formula

$$w'' = -w' \int_0^t (2w'(z, \tau)p'(w(z, \tau), \tau) + w(z, \tau)w'(z, \tau)p''(w(z, \tau), \tau)) d\tau,$$

which guarantees that w is C^3 on S^1 . Finally, we come to the conclusion that w is C^∞ on S^1 . \square

Let $f(z, t)$ denote $e^t w(z, t)$. The limit $\lim_{t \rightarrow \infty} f(z, t)$ is known [26] to be a representation of all univalent functions.

Let the driving term $p(z, t)$ in the Löwner-Kufarev ODE be from the Carathéodory class for almost all $t \geq 0$, C^∞ smooth in \hat{U} , and summable with respect to t . Then the domains $\Omega(t) = w(U, t)$ have smooth boundary $\partial\Omega(t)$. So the Löwner equation can be extended onto the closed unit disk $\hat{U} = U \cup S^1$.

Consider the Hamiltonian function given by

$$(10) \quad H = \int_{z \in S^1} f(z, t) (1 - p(e^{-t} f(z, t), t)) \bar{\psi}(z, t) \frac{dz}{iz},$$

on the unit circle $z \in S^1$, where $\psi(z, t)$ is a formal series

$$\psi(z, t) = \sum_{n=-k}^{\infty} \psi_n z^n,$$

defined about the unit circle S^1 for any $k \geq 0$. The Poisson structure on the symplectic space $(f, \bar{\psi})$ is given by the canonical brackets

$$\{P, Q\} = \frac{\delta P}{\delta f} \frac{\delta Q}{\delta \bar{\psi}} - \frac{\delta P}{\delta \bar{\psi}} \frac{\delta Q}{\delta f},$$

or in coordinate form (only ψ_n for $n \geq 1$ are independent co-vectors corresponding to the tangent vectors ∂_n with respect to the canonical Hermitean product for analytic functions)

$$\{p, q\} = \sum_{n=1}^{\infty} \frac{\partial p}{\partial c_n} \frac{\partial q}{\partial \bar{\psi}_n} - \frac{\partial p}{\partial \bar{\psi}_n} \frac{\partial q}{\partial c_n}.$$

Here

$$P(t) = \int_{z \in S^1} p(z, t) \frac{dz}{iz}, \quad Q(t) = \int_{z \in S^1} q(z, t) \frac{dz}{iz}.$$

The Hamiltonian system becomes

$$(11) \quad \frac{df(z, t)}{dt} = f(1 - p(e^{-t}f, t)) = \frac{\delta H}{\delta \bar{\psi}} = \{f, H\},$$

for the position coordinates and

$$(12) \quad \frac{d\bar{\psi}}{dt} = -(1 - p(e^{-t}f, t) - e^{-t}fp'(e^{-t}f, t))\bar{\psi} = \frac{-\delta H}{\delta f} = \{\bar{\psi}, H\},$$

for the momenta, where $\frac{\delta}{\delta f}$ and $\frac{\delta}{\delta \bar{\psi}}$ are the variational derivatives. So the phase coordinates $(f, \bar{\psi})$ play the role of the canonical Hamiltonian pair.

The coefficients c_n are the complex local coordinates on \mathcal{M} , so in these coordinates we have

$$\begin{aligned} \dot{c}_n &= \frac{dc_n}{dt} = c_n - \frac{e^t}{2\pi i} \int_{S^1} w(z, t) p(w(z, t), t) \frac{dz}{z^{n+2}}, \\ &= -\frac{1}{2\pi i} \int_{S^1} \sum_{k=1}^n e^{-kt} (e^t w)^{k+1} p_k \frac{dz}{z^{n+2}}, \quad n \geq 1. \end{aligned}$$

Let us fix some n and project the infinite dimensional Hamiltonian system on an n -dimensional \mathcal{M}_n . The dynamical equations for momenta governed by the Hamiltonian function (10) are

$$\dot{\bar{\psi}}_j = -\bar{\psi}_j + \frac{1}{2\pi i} \sum_{k=1}^n \bar{\psi}_k \int_{S^1} (p + wp') \frac{dz}{z^{k-j+1}}, \quad j = 1, \dots, n-1,$$

and

$$(13) \quad \dot{\bar{\psi}}_n = 0.$$

In particular,

$$\begin{aligned}\dot{c}_1 &= -e^{-t}p_1, \\ \dot{c}_2 &= -2e^{-t}p_1c_1 - e^{-2t}p_2, \\ \dot{c}_3 &= -e^{-t}p_1(2c_2 + c_1^2) - 3e^{-2t}p_2c_1 - e^{-3t}p_3, \\ \dots &\quad \dots\end{aligned}$$

for $n = 3$ we have

$$\begin{aligned}\dot{\bar{\psi}}_1 &= 2e^{-t}p_1\bar{\psi}_2 + (2e^{-t}p_1c_1 + 3e^{-2t}p_2)\bar{\psi}_3, \\ \dot{\bar{\psi}}_2 &= 2e^{-t}p_1\bar{\psi}_3, \\ \dot{\bar{\psi}}_3 &= 0.\end{aligned}$$

Let us set the function $L(z) := f'(z, t)\bar{\psi}(z, t)$. Let $(L(z))_{<0}$ mean the part of the Laurent series for $L(z)$ with negative powers of z ,

$$(L(z))_{<0} = (\bar{\psi}_1 + 2c_1\bar{\psi}_2 + 3c_2\bar{\psi}_3 + \dots)\frac{1}{z} + (\bar{\psi}_2 + 2c_1\bar{\psi}_3 + \dots)\frac{1}{z^2} + \dots = \sum_{k=1}^{\infty} \frac{L_k}{z^k}.$$

Then, the functions $L(z)$ and $(L(z))_{<0}$ are time-independent for all $z \in S^1$.

It is easily seen that, passing from the cotangent vectors $\bar{\psi}_k$ to the tangent vectors ∂_k , the coefficients L_k of $(L(z))_{<0}$ defined on the tangent bundle $T\mathcal{M}^{(1,0)}$ are exactly the Kirillov vector fields L_k . The corresponding fields \dot{L}_k in the covariant form are conserved by the Löwner-Kufarev ODE because $\dot{L}_k = \{L_k, H\} = 0$. The above Poisson structure coincides with that given by the Witt brackets introduced for L_k previously. For finite-dimensional grades this result was obtained in [22].

Let us formulate the result as a theorem.

Theorem 2. *Let the driving term $p(z, t)$ in the Löwner-Kufarev ODE be from the Carathéodory class for almost all $t \geq 0$, C^∞ smooth in \hat{U} , and summable with respect to t . Then the Kirillov fields in the covariant form are the conserved quantities for the Hamiltonian system (11–12) generated by the Löwner-Kufarev ODE.*

Remark 1. *Another way to construct a Hamiltonian system could be based on the symplectic structure given by the Kählerian form on $\text{Diff } S^1/S^1$. However, there is no explicit expression for such form in terms of functions $f \in \mathbf{S}$. Moreover, there must be a Hamiltonian formulation in which the Löwner-Kufarev equation becomes an evolution equation. This remains an open problem.*

Remark 2. *At a first glance the situation with an ODE with a parameter is quite simple. Indeed, if we solve an equation of type $\dot{f}(t, e^{i\theta}) = F(f(t, e^{i\theta}), t)$, then fixing θ we have an integral of motion $C = I(f(t, \cdot), t) = \text{const}$. Then, releasing θ , we have $C(e^{i\theta}) = I(f(t, e^{i\theta}), t)$. Expanding $C(e^{i\theta})$ into the Fourier series, we obtain an infinite number of conserved quantities, but they do not manifest an infinite number of degrees of freedom that govern the motion as in the field theory where the governing equations are PDE. In our case, we have not only one trajectory fixing the initial condition but a pencil of trajectories because our equation has an infinite number of control parameters, the Taylor coefficients of the function $p(z, t)$, which form a*

bounded non-linear set of admissible controls. Therefore, we operate with sections of the tangent and co-tangent bundles to the infinite dimensional manifold \mathcal{M} instead of vector fields along one trajectory as in usual ODE.

Remark 3. No linear combinations L_k^* of L_1, \dots, L_n, \dots allows us to reduce the system of $\{L_k\}$ to a new system of involutory $\{L_k^*\}$ in order to claim the Liouville integrability of our system. Observe that the coefficients in these linear combinations must be constants to keep conservation laws.

4.2. Construction of L_0 and L_{-n} . Consider again the generating function $L(z) = f'(z, t)\bar{\psi}(z, t)$ and the 'non-negative' part $(L(z))_{\geq 0}$ of the Laurent series for $L(z)$,

$$(L(z))_{\geq 0} = (\bar{\psi}_0 + 2c_1\bar{\psi}_1 + 3c_2\bar{\psi}_2 + \dots) + (\bar{\psi}_{-1} + 2c_1\bar{\psi}_0 + 3c_2\bar{\psi}_1 + \dots)z + \dots$$

$$= \sum_{k=0}^{\infty} \mathcal{L}_{-k} z^k.$$

All \mathcal{L}_{-k} are conserved by the construction. Define $\bar{\psi}_0^* = -\sum_{n=1}^{\infty} c_n \bar{\psi}_n$, and

$$L_0 = \mathcal{L}_0 - (\bar{\psi}_0 - \bar{\psi}_0^*).$$

The operator L_0 acts on the class \mathbf{S} by $L_0[f](z) = zf'(z) - f(z)$. Next define $L_{-1} = \mathcal{L}_{-1} - (\bar{\psi}_{-1} - \bar{\psi}_{-1}^*) - 2c_1(\bar{\psi}_0 - \bar{\psi}_0^*)$, where $\bar{\psi}_{-1}^* = 0$. Then,

$$L_{-1}[f](z) = f'(z) - 2c_1 f(z) - 1$$

Finally,

$$L_{-2} = \mathcal{L}_{-2} - (\bar{\psi}_{-2} - \bar{\psi}_{-2}^*) - 2c_1(\bar{\psi}_{-1} - \bar{\psi}_{-1}^*) - 3c_2(\bar{\psi}_0 - \bar{\psi}_0^*).$$

We choose $\bar{\psi}_{-2}^* = (c_3 - 3c_1c_2 + c_1^3)\bar{\psi}_1 + \dots$, so that

$$\bar{\psi}_{-2}^*[f](z) = \frac{1}{z} - \frac{1}{f(z)} - c_1 - (c_2 - c_1^2)f(z),$$

and

$$L_{-2}[f](z) = \frac{f'(z)}{z} - \frac{1}{f(z)} - 3c_1 + (c_1^2 - 4c_2)f(z).$$

An important fact is that

$$L_0 = c_1\bar{\psi}_1 + 2c_2\bar{\psi}_2 + \dots,$$

$$L_{-1} = (3c_2 - 2c_1^2)\bar{\psi}_1 + \dots,$$

$$L_{-2} = (5c_3 - 6c_1c_2 + 2c_1^3)\bar{\psi}_1 + \dots,$$

are linear with respect to $\bar{\psi}_k$, $k \geq 1$, and therefore, are sections of $T^*\mathcal{M}$, which are dual to Kirillov's vector fields. Equivalently,

$$L_{0,-1,-2}[f](z) = \text{function}(c_1, c_2, \dots)z^2 + \dots, \quad z^k = \frac{\partial f}{\partial c_{k-1}}.$$

All other co-vectors we construct by our Poisson brackets as

$$L_{-n} = \frac{1}{n-2}\{L_{-n+1}, L_{-1}\} = \frac{1}{n-4}\{L_{-n+2}, L_{-2}\}.$$

The form of the Poisson brackets guarantees us that all L_{-n} are linear with respect to $\bar{\psi}_1, \bar{\psi}_2, \dots$ and span the anti-holomorphic part of the co-tangent bundle $T^{(0,1)*}\mathcal{M}$.

Let us summarize the above in the following conclusion. We considered a non-linear contour dynamics given by the Löwner-Kufarev equation. It turned out to be underlined by an algebraic structure, namely, by the Witt algebra spanned by the Virasoro generators L_n , $n \in \mathbb{Z}$.

- L_n , $n = 1, 2, \dots$ are the holomorphic Virasoro generators. They span the holomorphic part of the complexified tangent bundle over the space of univalent functions, smooth on the boundary. In the covariant formulation they are conserved by the Löwner-Kufarev evolution.
- L_0 is the central element.
- L_{-n} , $n = 1, 2, \dots$ are the antiholomorphic Virasoro generators. They span the antiholomorphic part of the decomposition. They contain a conserved term and we give an iterative method to obtain them based on the Poisson structure of the Löwner-Kufarev evolution.

4.3. Löwner-Kufarev PDE. The Löwner equation in partial derivatives is

$$\dot{w}(\zeta, t) = \zeta w'(\zeta, t)p(\zeta, t), \quad \operatorname{Re} p(\zeta, t) > 0, \quad |\zeta| < 1.$$

with some initial condition $w(z, 0) = f_0(z)$. Let us consider the one-parameter family of functions $f(z, t) = e^{-t}w(z, t) = z(1 + \sum_{n=1}^{\infty} c_n(t)z^n)$, $f(z, 0) = f_0(z)$ as a C^1 path in \mathbf{S} . At the initial point $f_0(z)$ we have that $T_{f_0}\mathbf{S} = T_{f_0}\mathcal{M}^{(1,0)} = T_{f_0}\mathcal{M}$. A path in the coefficient body \mathcal{M} in the neighbourhood of f_0 is $(c_1(t), \dots, c_n(t), \dots)$ with the velocity vector $\dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots \in T_{f_0}\mathcal{M}$.

Taking the Virasoro generators $\{L_k\}$, $k \geq 1$, as a basis in $T_{f_0}\mathcal{M}^{(1,0)}$ we wish the velocity vector written in this new basis to be

$$(14) \quad \dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots = u_1L_1 + \dots u_nL_n + \dots$$

We compare (14) with the Löwner-Kufarev equation

$$(15) \quad \dot{f} = \dot{c}_1\partial_1 + \dots + \dot{c}_n\partial_n + \dots = zf'p(z, t) - f = L_0 + u_1L_1 + \dots u_nL_n + \dots,$$

where $p(z, t) = 1 + u_1z + \dots + u_nz^n + \dots$, and $L_0f = zf' - f$. In view of similarity between these two expressions (14) and (15), we notice that

- a new term L_0 appears in the Löwner-Kufarev equation;
- the function $p(z, t)$ with positive real part corresponds to subordination, whereas for generic trajectories it may have real part of arbitrary sign. We call this an *alternate Löwner-Kufarev evolution*;
- the vector L_0 corresponds exactly to the rotation:

$$e^{i\varepsilon}f(e^{-i\varepsilon}z) = f(z) - i\varepsilon(zf'(z) - f(z)) + o(\varepsilon).$$

Let us consider the set \mathbf{S}_0 of non-normalized smooth univalent functions of the form $F(z, t) = a_0(t)z + a_1(t)z^2 + \dots$, with a tangent vector $\dot{a}_0\partial_0 + \dots + \dot{a}_n\partial_n + \dots$, where $\partial_k = \partial/\partial a_k$, $k = 0, 1, 2, \dots$. Our aim is to define two different distributions for the tangent bundle $T\mathbf{S}_0$, that form a sub-bundle of co-dimension 1, which is the

tangent bundle $T\mathbf{S}$. This will be realized by means of formulas (14) and (15). Notice that $\partial_k F = z^{k+1}$. Setting $L_k(F) := z^{k+1}F'$ we get

$$\dot{F} = \dot{a}_0 \partial_0 + \dots + \dot{a}_n \partial_n + \dots = z f' p(z, t) = u_0 L_0 + u_1 L_1 + \dots u_n L_n + \dots,$$

where $p(z, t) = u_0 + u_1 z + \dots + u_n z^n + \dots$. This alternate Löwner-Kufarev equation represents recalculation of the tangent vector in the new basis

$$\dot{a}_0 \partial_0 + \dots + \dot{a}_n \partial_n + \dots = u_0 L_0 + \dots u_n L_n + \dots,$$

where $L_k = a_0 \partial_k + 2a_1 \partial_{k+1} + \dots$.

Let us present the distributions. We start with $F \in \mathbf{S}_0$, then we define $f \in \mathbf{S}$. The necessary distribution is the map

$$\mathbf{S}_0 \ni F \rightarrow T_f \mathbf{S} \hookrightarrow T_F \mathbf{S}_0.$$

The analytic form of the first distribution is the following factorization $f_1(z, t) = \frac{1}{a_0} F(z, t) = z + \frac{a_1}{a_0} z^2 + \dots$, so that

$$(16) \quad \dot{f}_1 = z f'_1 p(z, t) - \frac{\dot{a}_0}{a_0} f_1,$$

where $u_0 = \frac{\dot{a}_0}{a_0}$. Then we obtain

$$\dot{c}_1 \partial_1 + \dots + \dot{c}_n \partial_n + \dots = \hat{L}_0 + u_1 \hat{L}_1 + \dots + u_n \hat{L}_n + \dots$$

where $\hat{L}_0 f_1 = u_0(z f'_1 - f_1)$, $\hat{L}_k f_1 = z^{k+1} f'_1$, $c_k = \frac{a_k}{a_0}$, $\partial_k = \frac{\partial}{\partial c_k}$. In particular, $a_0 = e^t$ implies the Löwner-Kufarev equation for arbitrary sign of $\text{Re } p$.

The analytic form of the second distribution becomes $f_2(z, t) = F(\frac{1}{a_0} z, t) = z + \frac{a_1}{a_0} z^2 + \dots$, so that

$$(17) \quad \dot{f}_2 = z f'_2 p(\frac{z}{a_0}, t) - \frac{\dot{a}_0}{a_0} z f'_2,$$

where again $u_0 = \frac{\dot{a}_0}{a_0}$. In the coefficient form we get

$$\dot{c}_1 \partial_1 + \dots + \dot{c}_n \partial_n + \dots = u_1 \tilde{L}_1 + \dots + u_n \tilde{L}_n + \dots$$

where $\tilde{L}_k f_2 = z^{k+1} f'_2$, $c_k = \frac{a_k}{a_0^{k+1}}$, $\partial_k = \frac{\partial}{\partial c_k}$.

Observe that the equation (17) gives an identification of $T\mathcal{M}^{(1,0)}$ with $T\mathcal{M}$.

Finally, let us make an explicit calculation of \hat{L}_0 , which for $a_0 = e^t$ we continue to denote by L_0 . Using Kirilov's basis L_1, L_2, \dots as a linear combination we write

$$L_0 = \sum_{m=1}^{\infty} \Pi_m L_m.$$

The coefficients Π_m are polynomials, which can be obtained using the following recurrent formulas

$$K_1 = 0, \quad K_m = - \sum_{j=1}^{m-1} j(m-j+1) c_{m-j} c_j, \quad \Pi_m = m c_m + \sum_{j=1}^m K_{m-j+1} P_{j-1},$$

where P_k are polynomials

$$(18) \quad P_0 = 1, \quad P_1 = -2c_1, \quad P_2 = 4c_1^2 - 3c_2, \quad P_k = -\sum_{j=1}^k (j+1)c_j P_{k-j},$$

Let us summarize the above considerations in the following theorem.

Theorem 3. *The Löwner-Kufarev PDE (16) gives the distribution for the tangent bundle $T\mathbf{S}_0$ of non-normalized smooth univalent functions \mathbf{S}_0 , that forms a sub-bundle of co-dimension 1, which is the tangent bundle $T\mathbf{S}$.*

The equation (17) gives another distribution, and moreover, it makes the explicit correspondence between the natural complex structure of $T\mathbf{S}$, as \mathbf{S} embedded into $\mathbb{C}^{\mathbb{N}}$, and the complex structure of $T\mathcal{M}^{(1,0)}$ at each point $f \in \mathbf{S}$ defined by (3).

One of the reason to consider the alternate Löwner-Kufarev PDE is the regularized canonical Brownian motion on smooth Jordan curves. For all Sobolev metrics $H^{\frac{3}{2}+\varepsilon}$, the classical theory of stochastic flows allows to construct Brownian motions on C^1 diffeomorphism group of S^1 . The case $3/2$ is critical. Malliavin [21] constructed the canonical Brownian motion on the Lie algebra $\text{Vect } S^1$ for the Sobolev norm $H^{3/2}$. Another construction was proposed in [7]. Airault and Ren [2] proved that the infinitesimal version of the Brownian flow is Hölder continuous with any exponent $\beta < 1$.

The regularized canonical Brownian motion on $\text{Diff } S^1$ is a stochastic flow on S^1 associated to the Itô stochastic differential equation

$$dg_{x,t}^r = d\zeta_{x,t}^r(g_{x,t}^r),$$

$$\zeta_{x,t}^r(\theta) = \sum_{n=1}^{\infty} \frac{r^n}{\sqrt{n^3 - n}} (x_{2n}(t) \cos n\theta - x_{2n-1}(t) \sin n\theta),$$

where $\{x_k\}$ is a sequence of independent real-valued Brownian motions and $r \in (0, 1)$ and the series for $\zeta_{x,t}^r(\theta)$ is a Gaussian trigonometric series. Kunita's theory of stochastic flows asserts that the mapping $\theta \rightarrow g_{x,t}^r(\theta)$ is a C^∞ diffeomorphism and the limit $\lim_{r \rightarrow 1^-} g_{x,t}^r = g_{x,t}$ exists uniformly in θ . The random homeomorphism $g_{x,t}$ is called *canonical Brownian motion* on $\text{Diff } S^1$, see [2, 7, 21, 28]. It was shown in [2, 7], that this random homeomorphism is Hölder continuous.

The canonical Brownian motion can be defined not only on $\text{Diff } S^1$, but also on the space of C^∞ -smooth Jordan curves by conformal welding. This leads to dynamics of random loops which are not subordinated.

5. ELLIPTIC OPERATORS OVER THE COEFFICIENT BODY

The Kirillov first order differential operators L_k generate the elliptic operator $\sum |L_k|^2$. In this section we construct the geodesic equation and find geodesics with constant velocity coordinates in the field of this operator. In particular, we shall prove that the norm of the driving function in the Löwner-Kufarev theory gives the minimal energy of the motion in this field.

5.1. Dynamics within the coefficient body. Let us recall the geometry of the coefficient body \mathcal{M}_n for finite n . The affine coordinates are introduced by projecting

$$\mathcal{M} \ni f = z \left(1 + \sum_{k=1}^{\infty} c_k z^k \right) \mapsto (c_1, \dots, c_n) \in \mathcal{M}_n.$$

The manifold \mathcal{M}_n was studied actively in the middle of the last century, see e.g., [3, 30]. We compile some important properties of \mathcal{M}_n below:

- (i) \mathcal{M}_n is homeomorphic to a $(2n-2)$ -dimensional ball and its boundary $\partial\mathcal{M}_n$ is homeomorphic to a $(2n-3)$ -dimensional sphere;
- (ii) every point $x \in \partial\mathcal{M}_n$ corresponds to exactly one function $f \in \mathbf{S}$ which is called a *boundary function* for \mathcal{M}_n ;
- (iii) boundary functions map the unit disk U onto the complex plane \mathbb{C} minus piecewise analytic Jordan arcs forming a tree with a root at infinity and having at most n tips,
- (iv) with the exception for a set of smaller dimension, at every point $x \in \partial\mathcal{M}_n$ there exists a normal vector satisfying the Lipschitz condition;
- (v) there exists a connected open set X_1 on $\partial\mathcal{M}_n$, such that the boundary $\partial\mathcal{M}_n$ is an analytic hypersurface at every point of X_1 . The points of $\partial\mathcal{M}_n$ corresponding to the functions that give the extremum to a linear functional belong to the closure of X_1 .

Properties (ii) and (iii) imply that the functions from \mathbf{S} deliver interior points of \mathcal{M}_n . The Kirillov operators L_j restricted onto \mathcal{M}_n give truncated vector fields

$$L_j = \partial_j + \sum_{k=1}^{n-j} (k+1) c_k \partial_{j+k},$$

which we, if it causes no confusion, continue denoting by L_j in this section. In [22] based on the Löwner-Kufarev representation, we showed that these L_j can be obtained from a partially integrable Hamiltonian system for the coefficients in which the first integrals coincide with L_j .

Let $c(t) = (c_1(t), \dots, c_n(t))$ be a smooth trajectory in \mathcal{M}_n ; that is a C^1 map $c : [0, 1] \rightarrow \mathcal{M}_n$. Then the velocity vector $\dot{c}(t)$ written in the affine basis as $\dot{c}(t) = \dot{c}_1(t) \partial_1 + \dots + \dot{c}_n(t) \partial_n$ can be also represented in the basis of vector fields L_1, \dots, L_n (compare with (17)) as

$$\begin{aligned} (19) \quad \dot{c}(t) &= \dot{c}_1(t) \partial_1 + \dots + \dot{c}_n(t) \partial_n \\ &= u_1 L_1 + u_2 L_3 + \dots + u_n L_n, \end{aligned}$$

where the coefficients u_k can be written in the recurrent form as

$$(20) \quad u_1 = \dot{c}_1, \quad u_k = \dot{c}_k - \sum_{j=1}^{k-1} (j+1) \dot{c}_j u_{k-j}.$$

Expressing u_k in terms of c_k and \dot{c}_k , we get

$$(21) \quad u_k = \dot{c}_k + \sum_{j=1}^{k-1} P_j \dot{c}_{k-j}.$$

One may notice that these polynomials are the first coefficients of the holomorphic function $1/f'(z)$, where $f \in \mathbf{S}$. In the infinite dimensional case this follows from the Löwner-Kufarev equation (17) with $a_0 = e^t$. Kirillov's fields L_k act over these polynomials as

$$L_k P_n = (n - 2k - 1) P_{n-k} \quad n \geq k \quad \text{and} \quad L_k P_n = 0 \quad n < k.$$

Proposition 1. *We define*

$$(22) \quad \begin{aligned} \omega_1 &= dc_1, \\ \omega_2 &= dc_2 - 2c_1\omega_1, \\ &\dots \dots \dots, \\ \omega_n &= dc_n - \sum_{j=1}^{n-1} (j+1)c_j\omega_{n-j}. \end{aligned}$$

Then, $\{\omega_1, \dots, \omega_n\}$ is a conjugate to $\{L_1, \dots, L_n\}$ basis of one-forms. Namely,

$$\omega_n(L_n) = 1, \quad \omega_n(L_k) = 0 \quad \text{if } k \neq n.$$

Proof. If $k > n$, then the vector fields L_k do not contain ∂_n . Since the form ω_n depends only on dc_j with $j < n$, then

$$\omega_n(L_k) = \partial_n(L_k) - \sum_{j=1}^{n-1} (j+1)c_j\omega_{n-j}(L_k) = 0 \quad \text{for } k > n > n-j.$$

If $n = k$, then

$$\omega_n(L_n) = \partial_n(L_n) - \sum_{j=1}^{n-1} (j+1)c_j\omega_{n-j}(L_n) = 1 + 0 \quad \text{for } n > n-j.$$

To prove the case $k < n$ we apply the induction. Let us show for L_1 . We have

$$\omega_2(L_1) = dc_2(L_1) - 2c_1(L_1) = 2c_1 - 2c_1 = 0.$$

We suppose that $\omega_n(L_1) = 0$. Then

$$\omega_{n+1}(L_1) = dc_{n+1}(L_1) - \sum_{j=1}^n (j+1)c_j\omega_{n+1-j}(L_1) = (n+1)c_n - (n+1)c_n\omega_1(L_1) = 0.$$

The same arguments work for $\omega_n(L_k)$ with $k < n$. □

In the affine basis the forms ω_k can be written making use of the polynomials P_n . We observe that one-forms ω_k are defined in a similar way as the coordinates

u_k with respect to the Kirillov vector fields L_k . Thus, if we develop the recurrent relations (22) and collect the terms with dc_n we get

$$\omega_k = dc_k + \sum_{j=1}^{k-1} P_j dc_{k-j}. \quad k = 1, \dots, n.$$

By the duality of tangent and co-tangent bundles the information about the motion is encoded by these one-forms.

5.2. Hamiltonian equations. There exists an Hermitian form on $T\mathcal{M}_n$, such that the system $\{L_1, \dots, L_n\}$ is orthonormal with respect to this form. The operator $L = \sum |L_k|^2$ is elliptic, and we write the Hamiltonian function $H(c, \bar{c}, \psi, \bar{\psi})$ defined on the co-tangent bundle, corresponding to the operator L as $H(c, \bar{c}, \psi, \bar{\psi}) = \sum_{k=1}^n |l_k|^2$, where

$$l_k = \bar{\psi}_k + \sum_{j=1}^{n-k} (j+1) c_j \bar{\psi}_{k+j}.$$

The corresponding Hamiltonian system admits the form

$$\begin{aligned} \dot{c}_1 &= \frac{\partial H}{\partial \bar{\psi}_1} = \bar{l}_1 \\ \dots &= \dots \\ \dot{c}_n &= \frac{\partial H}{\partial \bar{\psi}_n} = \bar{l}_n + \sum_{j=1}^{n-1} (j+1) c_j \bar{l}_{n-j} \\ \dot{\bar{\psi}}_p &= -\frac{\partial H}{\partial c_p} = -(p+1) \sum_{k=1}^{n-p} l_k \bar{\psi}_{k+p} \\ \dots &= \dots \\ \dot{\bar{\psi}}_n &= -\frac{\partial H}{\partial c_n} = 0. \end{aligned}$$

Let us observe that

$$(23) \quad \dot{l}_k = \sum_{j=1}^{n-k} (j-k) \bar{l}_j l_{j+k}.$$

Expressing \bar{l}_k from the first n Hamiltonian equations we get

$$(24) \quad \bar{l}_k = \dot{c}_k + \sum_{j=1}^{k-1} P_j \dot{c}_{k-j}, \quad k = 1, \dots, n.$$

We can decouple the Hamiltonian system making use of (23) and (24) which leads us to the following non-linear differential equations of the second order

$$\ddot{c}_k = \dot{\bar{l}}_k + \sum_{j=1}^{k-1} (j+1) c_j \dot{\bar{l}}_{k-l} + \sum_{j=1}^{k-1} (j+1) \dot{c}_j \bar{l}_{k-l},$$

where \dot{l}_k are expressed in terms of the product of $\bar{l}_j l_{j+k}$ by (23), and the last products depend on P_j , \bar{P}_j and \dot{c} , \dot{c}_j for the corresponding indices j by (24). For example,

$$\ddot{c}_1 = \dot{\bar{l}}_1 = \sum_{j=1}^{n-1} (j-1) \left(\dot{c}_j + \sum_{p=1}^{j-1} P_p \dot{c}_{j-p} \right) \overline{\left(\dot{c}_{j+1} + \sum_{q=1}^j P_q \dot{c}_{j+1-q} \right)}.$$

Comparing (24) and (21), we conclude that $\bar{l}_k = u_k$ and u_k satisfy the differential equations

$$(25) \quad \dot{u}_k = \sum_{j=1}^{n-k} (j-k) \bar{u}_j u_{j+k},$$

on the solution of the Hamiltonian system. Observe that any solution of (25) has a velocity vector of constant length. It is easy to see from the following system

$$(26) \quad \begin{aligned} \bar{u}_1 \dot{u}_1 &= 0 \bar{u}_1 \bar{u}_1 u_2 + \bar{u}_1 \bar{u}_2 u_3 + 2 \bar{u}_1 \bar{u}_3 u_4 + 3 \bar{u}_1 \bar{u}_4 u_5 + 4 \bar{u}_1 \bar{u}_5 u_6 + \dots, \\ \bar{u}_2 \dot{u}_2 &= -1 \bar{u}_1 \bar{u}_2 u_3 + 0 \bar{u}_2 \bar{u}_2 u_4 + 1 \bar{u}_2 \bar{u}_3 u_5 + 2 \bar{u}_2 \bar{u}_4 u_6 + \dots, \\ \bar{u}_3 \dot{u}_3 &= -2 \bar{u}_1 \bar{u}_3 u_4 - 1 \bar{u}_2 \bar{u}_3 u_5 + 0 \bar{u}_3 \bar{u}_3 u_6 + \dots, \\ \bar{u}_4 \dot{u}_4 &= -3 \bar{u}_1 \bar{u}_4 u_5 - 2 \bar{u}_2 \bar{u}_4 u_6 + \dots, \\ \bar{u}_5 \dot{u}_5 &= -4 \bar{u}_1 \bar{u}_5 u_6 + \dots, \\ \bar{u}_6 \dot{u}_6 &= \dots \end{aligned}$$

Then,

$$\frac{d|u|^2}{dt} = 2 \sum_{k=1}^n (\bar{u}_k \dot{u}_k + u_k \dot{\bar{u}}_k) = 0,$$

for any n , thanks to the cut form of our vector fields and the skew symmetry of (26). The simplest solution may be deduced for constant driving terms u_k , $k = 1, \dots, n$. The Hamiltonian system immediately gives the geodesic

$$\begin{aligned} c_1 &= \bar{u}_1(0)s + c_1(0), \\ c_2 &= \bar{u}_1^2(0)s^2 + \bar{u}_2(0)s + c_2(0), \\ c_3 &= 3\bar{u}_1(0)\left(\bar{u}_1^2(0)\frac{s^3}{3} + \bar{u}_2(0)\frac{s^2}{2} + c_2(0)\right) + 2\bar{u}_2(0)\left(\bar{u}_1(0)\frac{s^2}{2} + c_1(0)s\right) + \bar{u}_3(0)s + c_3(0), \\ \dots &= \dots \end{aligned}$$

In general, c_n becomes a polynomial of order n with coefficients that depend on the initial data $c(0)$ and on the initial velocities $\bar{u}(0)$.

The Lagrangian \mathcal{L} corresponding to the Hamiltonian function H can be defined by the Legendre transform as

$$\mathcal{L} = (\dot{c}, \bar{\psi}) - H = \sum_{k=1}^n \left(\bar{l}_k \bar{\psi}_k + \bar{\psi}_k \sum_{j=1}^{k-1} (j+1) c_j \bar{l}_{k-j} \right) - \frac{1}{2} \sum_{k=1}^n |l_k|^2.$$

Taking into account that

$$\bar{\psi}_k \dot{c}_k = \sum_{j=1}^{k-1} (j+1) c_j \bar{\psi}_k \bar{l}_{k-j} + \bar{\psi}_k \bar{l}_k.$$

Summing up over k , we obtain $(\dot{c}, \bar{\psi}) = \sum_{k=1}^n l_k \bar{l}_k = \sum_{k=1}^n \bar{u}_k u_k$, that gives us

$$\mathcal{L}(c, \dot{c}) = \frac{1}{2} \sum_{k=1}^n |u_k|^2.$$

All these considerations can be generalized for $n \rightarrow \infty$. Thus, we conclude that the coefficients of the function $p(z, t)$ in the Löwner-Kufarev PDE play the role of generalized moments for the dynamics in \mathcal{M}_n and \mathcal{M} with respect to the Kirillov basis on the tangent bundle. Moreover, the L^2 -norm of the function p on the circle S^1 is the energy of such motion.

6. SLE AND CFT

In this section we briefly review for completeness the connections between conformal field theory (CFT) and Schramm-Löwner evolution (SLE) following, e.g., [4, 8]). SLE (being, e.g., a continuous limit of CFT's archetypical Ising model at its critical point) gives an approach to CFT which emphasizes CFT's roots in statistical physics.

SLE $_{\kappa}$ is a κ -parameter family of covariant processes describing the evolution of random sets called the SLE $_{\kappa}$ hulls. For different values of κ these sets can be either a simple fractal curve $\kappa \in [0, 4]$, or a self-touching curve $\kappa \in (4, 8)$, or a space filling Peano curve $\kappa \geq 8$. At this step we deal with the chordal version of SLE. The complement to a SLE $_{\kappa}$ hull in the upper half-plane \mathbb{H} is a simply connected domain that is mapped conformally onto \mathbb{H} by a holomorphic function $g(z, t)$ satisfying the equation

$$(27) \quad \frac{dg}{dt} = \frac{2}{g(z, t) - \xi_t}, \quad g(z, 0) = z,$$

where $\xi_t = \sqrt{\kappa} B_t$, and B_t is a normalized Brownian motion with the diffusion constant κ . The function $g(z, t)$ is expanded as $g(z, t) = z + \frac{2t}{z} + \dots$. The equation (27) is called the Schramm-Löwner equation and was studied first in [17]–[19], see also [29] for basic properties of SLE. Special values of κ correspond to interesting special cases of SLE, for example $\kappa = 2$ corresponds to the loop-erasing random walk and the uniform spanning tree, $\kappa = 4$ corresponds to the harmonic explorer and the Gaussian free field. Observe, that the equation (27) is not a stochastic differential equation (SDE). To rewrite it in a stochastic way (following [4], [8]) let us set a function $k_t(z) = g(z, t) - \xi_t$, where $k_t(z)$ satisfies already the SDE

$$dk_t(z) = \frac{2}{k_t(z)} dt - d\xi_t.$$

For a function $F(z)$ defined in the upper half-plane one can derive the Itô differential

$$(28) \quad dF(k_t) = -d\xi_t L_{-1} F(k_t) + dt \left(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2} \right) F(k_t),$$

with the operators $L_{-1} = -\frac{d}{dz}$ and $L_{-2} = -\frac{1}{z} \frac{d}{dz}$. These operators are the first two Virasoro generators in the ‘negative’ part of the Witt algebra spanned by the operators $-z^{n+1} \frac{d}{dz}$ acting on the appropriate representation space. All other generators can be obtained by the commutation relation

$$[L_m, L_n] = (n - m) L_{n+m}.$$

For any state $|\psi\rangle$, the state $L_{-1}|\psi\rangle$ measures the diffusion of $|\psi\rangle$ under SLE, and $(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2})|\psi\rangle$ measures the drift. The states of interest are drift-less, i.e., the second term in (28) vanishes. Such states are annihilated by $\frac{\kappa}{2} L_{-1}^2 - 2L_{-2}$, which is true if we choose the state $|\psi\rangle$ as the highest weight vector in the highest weight representation of the Virasoro algebra with the central charge c and the conformal weight h given by

$$c = \frac{(6 - \kappa)(3\kappa - 8)}{2\kappa}, \quad h = \frac{6 - \kappa}{2\kappa},$$

and the operators L_{-1} and L_{-2} are taken in the corresponding representation. It was obtained in [4] and [8], that $F(k_t)$ is a martingale if and only if $(\frac{\kappa}{2} L_{-1}^2 - 2L_{-2})F(k_t) = 0$. We define a CFT with a boundary in \mathbb{H} such that the boundary condition is changed by a boundary operator. The random curve in \mathbb{H} defined by SLE is growing so that it has states of one type to the left and of the other type to the right (the simplest way to view this is the lattice Ising model with the states defined as spin positions up or down). The mapping g satisfying (27) ‘unzips’ the boundary. The primary operator that induces the boundary change with the conformal weight h is drift-less, and therefore, its expectation value does not change in time under the boundary unzipping. Hence all correlators computing with this operator remain invariant. Analogous considerations one may provide for the ‘radial’ version of SLE in the unit disk, slightly modifying the above statements.

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